

# EXTENDIBILITY OF BILINEAR FORMS ON BANACH SEQUENCE SPACES

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ABSTRACT. We study Hahn-Banach extensions of multilinear forms defined on Banach sequence spaces. We characterize  $c_0$  in terms of extension of bilinear forms, and describe the Banach sequence spaces in which every bilinear form admits extensions to any superspace.

## 1. INTRODUCTION

One of the fundamental results in Functional Analysis is the Hahn-Banach theorem. It was proved independently by Hahn in 1927 [18] and by Banach in 1929 [5] (see also [6, Chapitre IV, §2]). In one of its forms, it states that if  $X$  is subspace of a normed space  $Z$ , then every continuous, linear functional  $f : X \rightarrow \mathbb{K}$  can be extended to  $Z$  preserving the norm. It soon became clear that a multilinear version of this result was not possible in general, and this started the search of situations on which such multilinear extension theorems are possible. A particular positive result was given by Arens in 1951, where he showed how to extend the product on a Banach algebra to its bidual and, also, how to extend bilinear operators defined on a couple of Banach spaces to their corresponding biduals [2, 3]. This is one of the lines to find extension theorems: given a space, find a superspace to which every multilinear mapping can be extended. Aron and Berner went further on this line and showed in 1978 that every holomorphic function on a Banach space can be extended to an open subset of the bidual [4].

Another line is to fix a Banach space  $X$  and consider the problem of extending bilinear forms defined on subspaces of  $X$ . Maurey's extension theorem [15, Corollary 12.23] is classical example of this natural point of view, in which relevant advances have been obtained in the last years [10, 26].

A third way to face the extension problem is to find the bilinear mappings (on a fixed Banach space) that can be extended to every superspace. This was the point of view taken by Grothendieck: in 1956 he showed in his théorème fondamental [17, page 60] that these are precisely those bilinear mappings factoring through Hilbert spaces via 2-summing operators. We say that a bilinear form  $T : X \times Y \rightarrow \mathbb{K}$  is *extendible* (see e.g. [7, 11, 20, 22]) if for all Banach spaces  $E \supset X$ ,  $F \supset Y$ , there exists a bilinear form defined on  $E \times F$  that extends  $T$ . Our aim, which can be framed in this last approach, is to describe those spaces which enjoy a bilinear (or multilinear) Hahn-Banach theorem, in the sense that every bilinear form is extendible. Examples of such spaces are  $A(\mathbb{D})$ ,  $H^\infty(\mathbb{D})$ ,  $\mathcal{L}^\infty$ -spaces and *Pisier* spaces, but a complete characterization is still unknown. In this line, our main result is the following theorem, which solves the problem among Banach spaces with unconditional basis.

**Theorem 1.1.** *The only Banach space with an unconditional basis on which every bilinear form is extendible is  $c_0$ .*

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This theorem will follow as a consequence of Theorem 2.2 below. We also characterize the Banach sequence spaces satisfying a bilinear Hahn-Banach theorem as those “between”  $c_0$  and  $\ell_\infty$  (see Corollary 2.4). As a byproduct, we obtain a partial answer to the following open problem: if a sequence  $X_n$  of  $n$ -dimensional Banach spaces is uniformly complemented in some  $\mathcal{L}_\infty$ , must these spaces be uniformly isomorphic to  $\ell_\infty^n$ ? Corollary 2.3 gives a positive answer for sections of a Banach sequence space (see Proposition 2.5).

**1.1. Preliminaries.** We briefly collect here some basic definitions that will be used throughout the paper. We will consider real or complex Banach spaces, that will be denoted  $X, Y, \dots$ . Unless otherwise stated they will be assumed to be infinite dimensional. The duals will be denoted by  $X^*, Y^*, \dots$ . Given two Banach spaces  $X$  and  $Y$ , we write  $X \approx Y$  if they are isomorphic and  $X \stackrel{1}{\approx} Y$  if they are isometrically isomorphic. We refer to [1, 25] for basic concepts and notations on Banach spaces.

We denote  $\mathcal{L}^2(X, Y)$  for the Banach space of all scalar valued, continuous, bilinear mappings (in short bilinear forms) on  $X \times Y$ . We write  $\mathcal{L}^2(X)$  whenever  $Y = X$ .

The space of extendible bilinear forms is denoted by  $\mathcal{E}^2(X, Y)$ . The *extendible norm*

$$\|T\|_{\mathcal{E}} = \|T\|_{\mathcal{E}^2(X, Y)} := \inf\{c > 0 : \text{for all } W \supseteq X, Z \supseteq Y \text{ there is an extension of } T \\ \text{to } W \times Z \text{ with norm } \leq c\}$$

makes  $\mathcal{E}^2(X, Y)$  a Banach space. Since every  $\ell_\infty(I)$  space is injective (in fact, has the metric extension property), every bilinear form on such spaces is extendible and the extendible and uniform norm coincide. Moreover, a bilinear form  $T$  on  $X \times Y$  is extendible if and only if it extends to  $\ell_\infty(I) \times \ell_\infty(J)$ , for some  $\ell_\infty(I) \supset X$  and  $\ell_\infty(J) \supset Y$ . The supremum defining the extendible norm can be taken only over the extensions to  $\ell_\infty(I) \times \ell_\infty(J)$ .

We write  $\mathcal{L}(X; Y)$  for the space of all (continuous, linear) operators  $u : X \rightarrow Y$ . We denote by  $\Pi_1(X; Y)$  the space of absolutely summing operators,  $\Gamma_\infty(X; Y)$  for the  $\infty$ -factorable and  $\Delta_2(X; Y)$  for the 2-dominated. Their corresponding norms are, respectively,  $\pi_1$ ,  $\gamma_\infty$  and  $\delta_2$  (see [13, 15] for definitions and basic properties).

We are going to use the theory tensor products and operator ideals as presented in [13]. We recall some notation and definitions for completeness. The projective tensor norm  $\pi$  is defined, for a  $z$  in the tensor product  $X \otimes Y$ , by

$$\pi(z) = \inf \left\{ \sum_{j=1}^r \|x_j\| \|y_j\| \right\},$$

where the infimum is taken over all the representations of  $z$  of the form  $z = \sum_{j=1}^r x_j \otimes y_j$ . The right-injective associate of  $\pi$  is denoted by  $\pi \setminus$ . This tensor norm is greatest right-injective tensor norm and makes the following inclusion an isometry:

$$X \otimes_{\pi \setminus} Y \stackrel{1}{\hookrightarrow} X \otimes_{\pi} \ell_\infty(B_{Y^*}),$$

where  $B_{Y^*}$  is the unit ball of  $Y^*$  (see [13, Theorem 20.7.] for details). Likewise, the injective associate  $/\pi \setminus$  is the largest injective tensor norm and is induced by the isometric inclusion

$$X \otimes_{/\pi \setminus} Y \stackrel{1}{\hookrightarrow} \ell_\infty(B_{X^*}) \otimes_{\pi} \ell_\infty(B_{Y^*}).$$

The metric extension property of  $\ell_\infty(I)$  spaces implies that extendible bilinear forms are precisely the  $/\pi\backslash$ -continuous ones:

$$\mathcal{E}^2(X, Y) \stackrel{1}{=} (X \otimes_{/\pi\backslash} Y)^*.$$

We refer to [13, 15] for all the basic (and not so basic) facts and any undefined notation on tensor norms and operator ideals.

Given a family  $\{X_n\}_n$  of Banach spaces where  $\dim X_n = n$ , we say that  $X_n$  are  $K$ -uniformly complemented in  $X$  if for each  $n$  we have a mapping  $i_n : X_n \rightarrow X$  and  $q_n : X \rightarrow X_n$  such that  $q_n \circ i_n$  is the identity on  $X_n$  and  $\|i_n\| \|q_n\| \leq K$ . In this case we also say that  $X$  contains  $X_n$  uniformly complemented. We note that if  $X$  contains uniform copies of  $\ell_\infty^n$  (i.e.,  $\mathbb{K}^n$  with the sup norm), then the  $\ell_\infty^n$  are uniformly complemented since they are injective spaces.

**1.2. Banach sequence spaces.** By a Banach sequence space (also known as Köthe sequence space) we will mean a Banach space  $X \subseteq \mathbb{K}^\mathbb{N}$  of sequences in  $\mathbb{K}$  such that  $\ell_1 \subseteq X \subseteq \ell_\infty$  with norm one inclusions satisfying that if  $x \in \mathbb{K}^\mathbb{N}$  and  $y \in X$  are such that  $|x_n| \leq |y_n|$  for all  $n \in \mathbb{N}$ , then  $x$  belongs to  $X$  and  $\|x\| \leq \|y\|$ .

If  $X$  is a Banach sequence space, we denote by  $\{e_n\}_n$  the sequence of canonical vectors, which is always a 1-unconditional basic sequence. We define  $X_N = \text{span}\{e_1, \dots, e_N\}$  and  $X_0 = \overline{\text{span}}\{e_n\}_n$ . This last space is usually referred to as the minimal kernel of  $X$ . Given  $x \in X$  we write  $x^N = (x_1, \dots, x_N)$ . There are inclusions  $i_N^X : X_N \hookrightarrow X$  and projections  $\pi_N^X : X \rightarrow X_N$  given by  $i_N^X(x_1, \dots, x_N) = (x_1, \dots, x_N, 0, 0, \dots)$  and  $\pi_N^X(x) = x^N$ . The inclusions are isometric and the projections have norm 1. For the case  $X = \ell_p$  ( $1 \leq p \leq \infty$ ), we write  $\ell_p^N$  for  $X_N$ .

Given a Banach sequence space  $X$ , its Köthe dual is defined as

$$X^\times = \{(z_n)_n \in \mathbb{K}^\mathbb{N} : \sum_n |z_n x_n| < \infty \text{ for all } x \in X\}.$$

With the norm  $\|z\|_{X^\times} = \sup_{\|x\|_X \leq 1} \sum_n |z_n x_n|$  it is again a Banach sequence space.

Following [24, 1.d], a Banach sequence space  $X$  is said to be  $r$ -convex (with  $1 \leq r < \infty$ ) if there exists a constant  $\kappa > 0$  such that for any choice  $x_1, \dots, x_N \in X$  we have

$$\left\| \left( \left( \sum_{j=1}^N |x_j(k)|^r \right)^{1/r} \right)_{k=1}^\infty \right\|_X \leq \kappa \left( \sum_{j=1}^N \|x_j\|_X^r \right)^{1/r}.$$

On the other hand,  $X$  is  $s$ -concave (with  $1 \leq s < \infty$ ) if there is a constant  $\kappa > 0$  such that

$$\left( \sum_{j=1}^N \|x_j\|_X^s \right)^{1/s} \leq \kappa \left\| \left( \left( \sum_{j=1}^N |x_j(k)|^s \right)^{1/s} \right)_{k=1}^\infty \right\|_X$$

for all  $x_1, \dots, x_N \in X$ .

It is well known that  $\ell_p$  is  $r$ -convex for  $1 \leq r \leq p$  and  $s$ -concave for  $p \leq s < \infty$ .

The following result is probably known. However we were not able to find a proper reference of this fact and we include here a short proof. It is modelled along the same lines as the proof of the fact that if the canonical vectors form a basis of  $X$  then both duals coincide.

**Proposition 1.2.** *If  $X$  is a Banach sequence space, its Köthe dual  $X^\times$  is a 1-complemented subspace of the usual dual  $X^*$ .*

*Proof.* Let us see first that the mapping  $i : X^\times \rightarrow X^*$  defined by  $i(z) = \varphi_z : X \rightarrow \mathbb{K}$ , with  $\varphi_z(x) = \sum_n z_n x_n$ , is an isometry. It is clearly well defined; moreover

$$\|\varphi_z\| = \sup_{x \in B_X} \left| \sum_n z_n x_n \right| \leq \sup_{x \in B_X} \sum_n |z_n x_n| = \|z\|_{X^\times}.$$

To see the reverse inequality, for any  $t, s \in \mathbb{K}$  we take  $\varepsilon(t, s) \in \mathbb{K}$  with  $|\varepsilon(t, s)| = 1$  such that  $|st| = \varepsilon(t, s)st$ ; then for every  $x \in B_X$  and every  $z \in X^\times$  we have

$$\sum_n |z_n x_n| = \sum_n \varepsilon(z_n, x_n) z_n x_n = \left| \sum_n \varepsilon(z_n, x_n) z_n x_n \right| \leq \sup_{a \in B_X} \left| \sum_n z_n a_n \right| = \|\varphi_z\|,$$

which gives  $\|z\|_{X^\times} \leq \|\varphi_z\|$ .

On the other hand, the mapping  $q : X^* \rightarrow X^\times$  given by  $q(\varphi) = (\varphi(e_n))_n$  defines a norm-one projection. Indeed, given  $x \in X$  and fixed  $N$  we have

$$\begin{aligned} \sum_{n=1}^N |x_n \varphi(e_n)| &= \sum_{n=1}^N \varepsilon(x_n, \varphi(e_n)) x_n \varphi(e_n) = \varphi \left( \sum_{n=1}^N \varepsilon(x_n, \varphi(e_n)) x_n e_n \right) \\ &\leq \|\varphi\| \left\| \sum_{n=1}^N \varepsilon(x_n, \varphi(e_n)) x_n e_n \right\|_X \leq \|\varphi\| \|x\|. \end{aligned}$$

This shows that  $\sum_{n=1}^\infty |x_n \varphi(e_n)| \leq \|\varphi\| \|x\|$ , which gives that  $q$  is well defined and  $\|q(\varphi)\| \leq \|\varphi\|$ . Furthermore,  $q$  is a projection, since clearly  $q \circ i(z) = q(\varphi_z) = (z_n)_n = z$ .  $\square$

## 2. EXTENSION OF BILINEAR FORMS ON BANACH SEQUENCE SPACES

In what follows  $K_G$  denotes the Grothendieck's constant. We begin by proving the following known fact, which was stated as Theorem 3.4 in [20] without the estimates for the norms (see [11, Lemma 2.4] for a result in the same spirit).

**Proposition 2.1.** *If every bilinear form  $B : X \times Y \rightarrow \mathbb{K}$  is extendible with  $\|B\|_{\mathcal{E}} \leq K\|B\|$ , then every operator  $u : X^* \rightarrow \ell_2$  is absolutely 1-summing and  $\pi_1(u) \leq K_G K \|u\|$ .*

*Proof.* We first note that, by definition of the tensor norm  $\wedge/\pi\backslash$  (see [13, Section 20.7]),  $\mathcal{E}^2(X, Y)$  is isometrically the dual of  $X \otimes_{\wedge/\pi\backslash} Y$ . Then, our hypothesis is equivalent to the inequality  $\pi \leq K/\pi\backslash$  on  $X \otimes Y$  and, as a consequence, we also have  $\pi\backslash \leq K/\pi\backslash$  on  $X \otimes Y$ . Since both  $\pi\backslash$  and  $\wedge/\pi\backslash$  are right-injective, an application of Dvoretzky's theorem [15, 19.1] and the previous inequality gives an isomorphism

$$(1) \quad X \otimes_{\wedge/\pi\backslash} \ell_2^N \longrightarrow X \otimes_{\pi\backslash} \ell_2^N$$

with norm at most  $K$ . Since  $\ell_2^N$  is finite dimensional,  $\mathcal{L}(X; \ell_2^N)$  and  $X^* \otimes \ell_2^N$  coincide as sets. Then, the embedding in [13, Section 17.6] is actually surjective and [13, Sections 21.5 and 27.2] give  $X^* \otimes_{w_\infty} \ell_2^N \stackrel{1}{\approx} \Gamma_\infty(X; \ell_2^N)$  (see [15, Chapter 9] or [13, Section 18] for the definition of  $\Gamma_\infty(X; Y)$ ). Therefore,

$$(2) \quad (X \otimes_{\pi\backslash} \ell_2^N)^{**} \stackrel{1}{\approx} (\Gamma_\infty(X, \ell_2^N))^* \stackrel{1}{\approx} (X^* \otimes_{w_\infty} \ell_2^N)^* \stackrel{1}{\approx} \Pi_1(X^*; \ell_2^N).$$

Now, by [13, Sections 17.12 and 27.2], the operator ideal  $\Delta_2$  is associated to  $w_2$  and  $\Pi_1$  is associated to  $\pi\backslash$ . On the other hand Grothendieck's inequality [13, Section 20.17] states that

$w_2 \geq K_G/\pi$  and clearly we have  $\mathcal{L}(X^*; \ell_2^N) \stackrel{1}{\approx} \Delta_2(X^*; \ell_2^N)$ . Using (2) to take biduals in (1) we have an isomorphism

$$(3) \quad \mathcal{L}(X^*; \ell_2^N) \longrightarrow (X \otimes_{\pi} \ell_2^N)^{**} \longrightarrow (X^{**} \otimes_{\pi} \ell_2^N)^{**} \stackrel{1}{\approx} \Pi_1(X^*; \ell_2^N),$$

where the first mapping has norm bounded by  $K_G$  and the second one by  $K$ . Since both  $\mathcal{L}$  and  $\Pi_1$  are maximal operator ideals, the same holds if we put  $\ell_2$  instead of  $\ell_2^N$ .  $\square$

With this result we can now prove the following one, from which Theorem 1.1 follows as an immediate consequence.

**Theorem 2.2.** *Let  $X$  be a Banach space with an unconditional basis and  $Y$  be any infinite dimensional Banach space such that every bilinear form on  $X \times Y$  is extendible. Then  $X \approx c_0$ .*

*Proof.* Let us see first that, under our assumptions, the basis of  $X$  must be shrinking. Suppose it is not. Since it is unconditional, by James theorem [19, Corollary 2] (see also [1, Theorem 3.3.1])  $X$  must contain a complemented copy of  $\ell_1$ . Since the property of all bilinear forms being extendible is inherited by complemented subspaces, it follows that every bilinear form on  $\ell_1 \times Y$  is extendible. This implies [22, Lemma 6] that every continuous linear operator from  $Y$  to  $\ell_\infty$  is absolutely 2-summing. By the so called  $\mathfrak{L}_p$ -Local Technique Lemma for Operator Ideals [13, Section 23.1], the same holds for every operator from  $Y$  to  $\ell_\infty(I)$ , for any index set  $I$ . But this is not possible, since there exists an isometric embedding from  $Y$  into some  $\ell_\infty(I)$ , and this cannot be absolutely 2-summing (otherwise, the identity on  $Y$  would be so, but  $Y$  is infinite dimensional).

This means that the canonical basis of  $X$  must be shrinking. We can assume that the basis is 1-unconditional, so that the coordinate basis is an 1-unconditional basis of  $X^*$ . We also know from Proposition 2.1 that all operators from  $X^*$  to  $\ell_2$  are absolutely 1-summing. By [23] (see also [25, Theorem 8.21]) this implies that the basis of  $X^*$  is  $(K_G K)^2$ -equivalent to the basis of  $\ell_1$ . Although  $\ell_1$  has many non-isomorphic preduals, if the coordinate basis is equivalent to that of  $\ell_1$ , a standard computation shows that the canonical basis on  $X$  must be  $(K_G K)^2$ -equivalent to the basis of  $c_0$ .  $\square$

Note that the proof not only shows that  $X$  must be isomorphic to  $c_0$ , but also gives an estimation of the Banach-Mazur distance between  $X$  and  $c_0$  whenever  $X$  has an 1-unconditional basis. As a consequence, we can also characterize the pairs of Banach sequence spaces on which every bilinear form is extendible.

**Corollary 2.3.** *If  $X$  and  $Y$  are Banach sequence spaces, then the following are equivalent.*

- (i)  $\mathcal{L}^2(X, Y) = \mathcal{E}^2(X, Y)$  and  $\|B\|_{\mathcal{E}} \leq K_1 \|B\|$  for all bilinear form  $B$  on  $X \times Y$ .
- (ii) The canonical basic sequence of  $X$  and  $Y$  are  $K_2$ -equivalent to the canonical basis of  $c_0$ .
- (iii)  $K_3 := \sup\{d(X_N, \ell_\infty^N), d(Y_N, \ell_\infty^N) : N \in \mathbb{N}\}$  is finite (where  $d$  denotes the Banach-Mazur distance).
- (iv) The spaces  $X_N$  and  $Y_N$  ( $N \in \mathbb{N}$ ) are  $K_4$ -uniformly complemented in some  $L_\infty(\mu)$ .

Moreover, we have  $K_4 \leq K_3 \leq K_2 \leq (K_G K_1)^2$  and  $K_1 \leq K_2^2 \leq K_G^4 K_4^8$ .

*Proof.* If (i) holds on  $X \times Y$ , then the same holds for  $X_N \times Y_N$  for any  $N$  and, by the density lemma [13, Section 13.4], for  $X_0 \times Y_0$ . By Theorem 2.2, both bases are  $(K_G K)^2$ -equivalent to the basis of  $c_0$ .

The implications (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (iv) are immediate, as well as the inequalities  $K_4 \leq K_3 \leq K_2$ .

If (iv) holds, bilinear forms on  $X_N \times Y_N$  are extendible with  $\|\cdot\|_{\mathcal{E}} \leq K_4^2 \|\cdot\|$  and, as before, the same holds for  $X_0 \times Y_0$ . By Theorem 2.2 their canonical bases are  $K_G^2 K_4^4$ -equivalent to the canonical basis of  $c_0$ , which is (ii).

Now suppose (ii) holds and take a bilinear form  $B : X \times Y \rightarrow \mathbb{K}$ . We know from Proposition 1.2 that  $X^\times$  is 1-complemented in  $X^*$ . Then  $(X^\times)^*$  is isometrically a (complemented) subspace of  $X^{**}$ . Since (ii) implies  $X^\times = \ell_1$ , we also have  $(X^\times)^* = X^{\times\times} = \ell_\infty$ . The same holds for  $Y$ , so we obtain the following diagrams:

$$\begin{array}{ccc} X & \xrightarrow{i_1} & X^{\times\times} \xrightarrow{i_2} X^{**} \\ & \uparrow u & \\ & \ell_\infty & \end{array} \quad \begin{array}{ccc} Y & \xrightarrow{j_1} & Y^{\times\times} \xrightarrow{j_2} Y^{**} \\ & \uparrow v & \\ & \ell_\infty & \end{array},$$

where  $i_1, i_2, j_1$  and  $j_2$  are isometric injections and  $u$  and  $v$  are isomorphisms with

$$(4) \quad \|u\| \|u^{-1}\| \leq K_2 \quad \text{and} \quad \|v\| \|v^{-1}\| \leq K_2.$$

We can extend  $B$  (in the canonical way) to a bilinear form  $\tilde{B} : X^{**} \times Y^{**} \rightarrow \mathbb{K}$  with the same norm as  $B$ , and then define a bilinear form  $\hat{B}$  on  $\ell_\infty \times \ell_\infty$  by  $\hat{B} = \tilde{B} \circ (i_2 \circ u, j_2 \circ v)$ . We have obtained the factorization  $B = \hat{B} \circ (u^{-1} \circ i_1, v^{-1} \circ j_1)$ . Since on  $\ell_\infty \times \ell_\infty$  every bilinear form is extendible (with the extendible norm equal to the usual norm), from the ideal property of extendible bilinear forms and inequalities (4) we conclude that  $B$  is extendible and  $\|B\|_{\mathcal{E}} \leq K_2^2 \|B\|$ .  $\square$

It follows from the previous corollary (and its proof) that a Banach sequence space on which every bilinear form is extendible must satisfy the sublattice inclusions

$$(5) \quad c_0 \subset X \subset \ell_\infty.$$

Conversely, let  $X$  be a Banach sequence space satisfying (5). By a closed graph argument, both inclusions are continuous and it is easy to check that  $X$  satisfies the equivalent conditions of Corollary 2.3. Note also that a Banach sequence space satisfies (5) if and only if its Köthe dual is  $\ell_1$ . As a consequence, we have the following version of Theorem 1.1 for Banach sequence spaces.

**Corollary 2.4.** *The Banach sequence spaces  $X$  on which every bilinear form is extendible are those satisfying (5). Also, this happens if and only if  $X^\times = \ell_1$ .*

Examples of such spaces are  $c_0 \oplus \ell_\infty$ ,  $c_0(\ell_\infty)$  and  $\ell_\infty(c_0)$ . It is not hard to see that these spaces are mutually non-isomorphic Banach sequence spaces (see also [12], where the authors show that  $c_0(\ell_\infty)$  and  $\ell_\infty(c_0)$  are not isomorphic even as Banach spaces).

If a sequence  $X_n$  of  $n$ -dimensional Banach spaces is uniformly complemented in some  $\mathcal{L}_\infty$ , it is an open problem if these spaces have to be uniformly isomorphic to  $\ell_\infty^n$ . Taking  $X = Y$  in Corollary 2.3, the implication (iv)  $\Rightarrow$  (iii) gives the following partial answer.

**Proposition 2.5.** *If the  $N$ -dimensional sections  $X_N$  of a Banach sequence space  $X$  are uniformly complemented in some  $L_\infty(\mu)$ , then they must be uniformly isomorphic to  $\ell_\infty^N$ .*

Note also that, since  $\ell_\infty$  and  $c_0$  are the only symmetric Banach sequence spaces satisfying (ii) of Corollary 2.3, these two are the only symmetric Banach sequence spaces on which every bilinear form is extendible.



If  $X_1, \dots, X_n$  are Banach spaces such that every  $n$ -linear form on  $X_1 \times \dots \times X_n$  is extendible, then it is known (and easy to see) that so is every bilinear form on  $X_i \times X_j$  for each pair  $i \neq j$ . Indeed, given  $B \in \mathcal{L}^2(X_i \times X_j)$ , we can multiply it by linear functionals to obtain a  $n$ -linear form on  $X_1 \times \dots \times X_n$ . This is extendible by our hypothesis. From this, it is rather immediate to conclude that  $B$  is extendible. As a consequence, multilinear versions of our results follow directly from the bilinear ones.

If  $X$  and  $Y$  are Banach sequence spaces such that every bilinear form on  $X \times Y$  is extendible, then we know from Theorem 2.3 (iii) that both  $X$  and  $Y$  contain the  $\ell_\infty^N$  uniformly. We can extend this statement to subspaces of Banach lattices.

**Proposition 2.6.** *Let  $X_1, X_2$  be subspaces of Banach lattices such that every  $n$ -linear form on  $X_1, X_2$  is extendible. Then every infinite dimensional complemented subspace of each  $X_j$  contains the  $\ell_\infty^N$  uniformly.*

*Proof.* Suppose that there exists a complemented subspace  $E$  of  $X_1$  that does not contain the  $\ell_\infty^N$  uniformly. By [21, Corollary 1],  $E$  must contain uniformly complemented  $N$ -dimensional subspaces  $E_N$  such that  $\sup_N d(E_N, \ell_p^N) < \infty$  for  $p = 1$  or  $2$ . Since  $E$  is complemented in  $X_1$ , the  $E_N$  are also uniformly complemented in  $X_1$ . On the other hand, again by [21, Corollary 1],  $X_2$  must contain uniformly complemented  $N$ -dimensional subspaces  $F_N$  such that  $\sup_N d(F_N, \ell_q^N) < \infty$  for  $q = 1, 2$  or  $\infty$ . Our hypotheses ensure that bilinear forms on  $X_1 \times X_2$  are extendible. Since  $E_N$  and  $F_N$  are uniformly complemented in  $X_1$  and  $X_2$  and they are (uniformly) isomorphic to  $\ell_p^N$  and  $\ell_q^N$ , there must exist  $K > 0$  such that  $\|B\|_{\mathcal{L}^2(\ell_p^N, \ell_q^N)} \leq K\|B\|_{\mathcal{L}^2(\ell_p^N, \ell_q^N)}$  for all  $N$ . Now, the density lemma [13, Section 13.4] implies that every bilinear form on  $\ell_p \times \ell_q$  (or  $\ell_p \times c_0$ ) must be extendible, which contradicts Theorem 2.2.  $\square$

The converse of Proposition 2.6 does not hold. For example, the Schreier space is  $c_0$ -saturated and there are non-extendible bilinear forms on it (since there are bilinear forms which are not weakly sequentially continuous). Another counterexample of the converse is  $d_*(w, 1)$ , the predual of the Lorentz sequence space  $d(w, 1)$  (see [27] or [16] for a description of the predual). Since these examples are Banach sequence spaces, they also show that assertion (iii) in Theorem 2.3 is strictly stronger than containing  $\ell_\infty^N$  uniformly.

In Banach sequence spaces, diagonal bilinear forms are the simplest ones. These are the bilinear forms  $T_\alpha : X_1 \times X_2 \rightarrow \mathbb{C}$  given by

$$T_\alpha(x^1, x^2) = \sum_{k=1}^{\infty} \alpha_k x_k^1 x_k^2,$$

for some sequence  $(\alpha_k)_k$  of scalars. We end this note showing, under some assumptions, which are the spaces on which all diagonal bilinear forms are extendible.

Following standard notation, given a symmetric Banach sequence space  $X$  we consider the fundamental function of  $X$ , given by  $\lambda_X(N) := \left\| \sum_{k=1}^N e_k \right\|_X$  for  $N \in \mathbb{N}$ .

Given two sequence of real numbers  $(a_n)_n$  and  $(b_n)_n$  we write  $a_n \preceq b_n$  whenever there is a universal constant  $C > 0$  such that  $a_n \leq Cb_n$  for every  $n$ . If  $a_n \preceq b_n$  and  $b_n \preceq a_n$ , we write  $a_n \asymp b_n$ .

**Theorem 2.7.** *Let  $X$  and  $Y$  be symmetric Banach sequence spaces, each being 2-convex or 2-concave. Then all diagonal bilinear forms on  $X \times Y$  are extendible if and only if either  $X = Y = \ell_1$  or  $X, Y \in \{c_0, \ell_\infty\}$*

*Proof.* The *if* part is clear: by [8, Proposition 2.3] (see also [9, Proposition 1.2]) on  $\ell_1$  diagonal bilinear forms are integral (and, therefore, extendible), and in the other cases all bilinear forms are extendible.

For the converse, we consider the diagonal bilinear form given by  $\phi_N(x, y) = \sum_{i=1}^N x_i y_i$ . It is easily computed that  $\|\phi_N\|_{\mathcal{L}^2(\ell_2^N)} = 1$ ; on the other hand, by [8, Proposition 1.1] or [11, Proposition 2.5] we have  $\|\phi_N\|_{\mathcal{E}^2(\ell_2^N)} = \|\phi_N\|_{\mathcal{N}^2(\ell_2^N)} = N$ . Let now  $\text{id}_X^N : \ell_2^N \rightarrow X_N$  and  $\text{id}_Y^N : \ell_2^N \rightarrow Y_N$  be the identity mappings. Comparing the usual and extendible norms of the bilinear forms  $\phi_N$  and  $\phi_N \circ ((\text{id}_X^N)^{-1}, (\text{id}_Y^N)^{-1})$ , we get

$$N \preceq \|\text{id}_X^N\| \|\text{id}_Y^N\| \|(\text{id}_X^N)^{-1}\| \|(\text{id}_Y^N)^{-1}\|.$$

By [28, 16.4] (see also [14, page 138]), since  $X$  is a symmetric Banach sequence space we have  $d(\ell_2^N, X_N) = \|\text{id}_X^N\| \|(\text{id}_X^N)^{-1}\|$  (and the same for  $Y_N$ ). Therefore,

$$N \preceq \|\text{id}_X^N\| \|\text{id}_Y^N\| \|(\text{id}_X^N)^{-1}\| \|(\text{id}_Y^N)^{-1}\| = d(\ell_2^N, X_N) d(\ell_2^N, Y_N).$$

Since we always have  $d(\ell_2^N, X_N) \leq \sqrt{N}$  and  $d(\ell_2^N, Y_N) \leq \sqrt{N}$ , we can conclude that  $\sqrt{N} \asymp d(\ell_2^N, X_N) = \|\text{id}_X^N\| \|(\text{id}_X^N)^{-1}\|$  (and the same for  $Y_N$ ). We now apply [14, Lemma 1 (i)] and get:

$$\max\left(\frac{1}{\lambda_X(N)}, \frac{\lambda_X(N)}{N}\right) \asymp 1.$$

From this we can conclude that  $X$  must be  $\ell_1, c_0$  or  $\ell_\infty$ . Indeed, suppose we split the natural numbers  $\mathbb{N} = I \cup J$ , so that  $(\frac{1}{\lambda_X(N)})_{N \in I} \asymp 1$  and  $(\frac{\lambda_X(N)}{N})_{N \in J} \asymp 1$ . We have then that  $(\lambda_X(N))_{N \in I}$  is bounded and  $(N)_{N \in J} \preceq (\lambda_X(N))_{N \in J}$ . Since  $(\lambda_X(N))_{N \in \mathbb{N}}$  is non-decreasing, either  $I$  or  $J$  must be finite. If  $J$  is finite, then  $(\lambda_X(N))_{N \in \mathbb{N}}$  is bounded and then the norm in  $X$  is equivalent to the sup norm and  $X$  is  $c_0$  or  $\ell_\infty$ . If  $I$  is finite, then  $N \preceq (\lambda_X(N))_{N \in \mathbb{N}}$ . Although the fundamental sequence of a symmetric Banach sequence space does not characterize the norm, for this extreme case it is possible to prove that the norm on  $X$  must be isomorphic to  $\ell_1$ : from the estimate  $N \preceq (\lambda_X(N))_{N \in \mathbb{N}}$  we easily obtain  $\lambda_{X^\times}(N) \asymp 1$  and, by the previous case,  $X^\times$  must be  $\ell_\infty$ . Then we have  $X = \ell_1$ . Proceeding in the same way,  $Y$  has to be either  $\ell_1, c_0$  or  $\ell_\infty$ .

It remains to show that on  $c_0 \times \ell_1$  and on  $\ell_1 \times \ell_\infty$  there are non-extendible diagonal bilinear forms. The mapping  $c_0 \times \ell_1$  given by  $(x, x') \mapsto x'(x)$  is the diagonal bilinear form induced by the formal identity. An extension of this mapping to  $c_0 \times \ell_\infty$  would give a projection from  $\ell_\infty$  to  $\ell_1$  (see [13, 1.5]), which does not exist. For  $\ell_1 \times \ell_\infty$  we can reason in a similar way.  $\square$

Both assumptions on symmetry and concavity/convexity in “only if” part of the previous theorem cannot be omitted. Indeed, if we take  $c_0 \oplus \ell_1$  (that seen as a sequence space is neither symmetric nor 2-concave or 2-convex), then every diagonal bilinear form on  $(c_0 \oplus \ell_1) \times (c_0 \oplus \ell_1)$  is the sum of a diagonal bilinear form on  $c_0 \times c_0$  and a diagonal bilinear form on  $\ell_1 \times \ell_1$ , and is therefore extendible.

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